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On bipartite distance-regular graphs with a strongly closed subgraph of diameter three

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ABSTRACT

Let Γ denote a distance-regular graph with a strongly closed regular subgraph Y. Hosoya and Suzuki [R. Hosoya, H. Suzuki, Tight distance-regular graphs with respect to subsets, European J. Combin. 28 (2007) 61–74] showed an inequality for the second largest and least eigenvalues of Γ in the case Y is of diameter 2. In this paper, we study the case when Γ is bipartite and Y is of diameter 3, and obtain an inequality for the second largest eigenvalue of Γ. Moreover, we characterize the distance-regular graphs with a completely regular strongly closed subgraph $H(3, 2)$.

1. Introduction

Let $\Gamma = (X, R)$ denote a simple connected graph with the vertex set X and the edge set R. For vertices x and y, $\partial(x, y)$ denotes the *distance* between x and y, i.e., the length of a shortest path connecting x and y. Let $d := \max\{\partial(x, y) | x, y \in X\}$ denote the *diameter* of Γ. For $x \in X$ and $i \in \{0, 1, \ldots, d\}$, define

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

For two vertices x and y at distance i, define

$$C(x, y) = \Gamma_{i-1}(x) \cap \Gamma_1(y),$$
$$A(x, y) = \Gamma_i(x) \cap \Gamma_1(y),$$

and

$$B(x, y) = \Gamma_{i+1}(x) \cap \Gamma_1(y),$$

where $\Gamma_{-1}(x) = \Gamma_{d+1}(x) = \emptyset$.

A graph Γ is said to be *distance-regular* if the cardinalities $c_i = |C(x, y)|$, $a_i = |A(x, y)|$, and $b_i = |B(x, y)|$ are constants whenever $\partial(x, y) = i$ $(i = 0, 1, \ldots, d)$.

Let Y be a non-empty subset of X. When there is no danger of confusion, we also use Y to denote the induced subgraph on it. Y is called a *strongly closed* subgraph of Γ if $C(x, y) \cup A(x, y) \subseteq Y$ whenever $x, y \in Y$. For $x \in X$, define

$$\partial(x, Y) = \min\{\partial(x, y) | y \in Y\}.$$

The numbers

$$\tau = \tau(Y) = \max\{\partial(x, Y) | x \in X\}, \qquad w(Y) = \max\{\partial(x, y) | x, y \in Y\}$$

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are called the *covering radius*, the *width* of Y in Γ, respectively. For $i \in \{0, 1, \dots, \tau\}$, let

$$\Gamma_i(Y) = \{x \in X | \partial(x, Y) = i\}.$$

A subgraph Y is called *completely regular* if for any integer i,

$$\gamma_i = |\Gamma_{i-1}(Y) \cap \Gamma_1(x)|, \qquad \alpha_i = |\Gamma_i(Y) \cap \Gamma_1(x)|, \qquad \beta_i = |\Gamma_{i+1}(Y) \cap \Gamma_1(x)|$$

are constants whenever $x \in \Gamma_i(Y)$.

Take Ω to be a finite set of cardinality $q \geq 2$. The *Hamming graph* $H(d, q)$ has vertex set Ω^d, the Cartesian product of d copies of Ω; two vertices are adjacent whenever they differ in exactly one coordinate.

Let Γ denote a distance-regular graph with a regular subgraph Y of diameter 2. Hosoya and Suzuki [3] proved an inequality for the second largest and least eigenvalues of Γ. In this paper, we study the case when Γ is bipartite and Y is of diameter 3, and obtain the following results:

Theorem 1.1. *Let $\Gamma = (X, R)$ be a bipartite distance-regular graph with a strongly closed regular subgraph Y of diameter 3. Then*

$$\theta_1 \leq \frac{b_2}{\sqrt{c_3 - c_2}}, \tag{1}$$

where θ_1 is the second largest eigenvalue of Γ.

Proposition 1.2. *Let $\Gamma = (X, R)$ be a distance-regular graph with a strongly closed subgraph $Y = H(3, 2)$. If Y is completely regular with covering radius $d - 3$, then Γ is isomorphic to $H(d, 2)$.*

2. Preliminaries

In this section we recall some basic concepts about distance-regular graphs. Our notation and terminologies are standard. The reader is referred to [1,2] for more information.

In the rest of this paper, we always assume that $\Gamma = (X, R)$ is a distance-regular graph of diameter d on n vertices. For each i, let A_i denote the matrix with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i, \\ 0, & \text{otherwise.} \end{cases}$$

We abbreviate $A = A_1$. The subalgebra $\mathcal{M}$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by A is call the *Bose–Mesner algebra* of Γ, and $A_0, A_1, \dots, A_d$ form a basis of $\mathcal{M}$. It is well known that $\mathcal{M}$ has a basis consisting of *primitive idempotents* $E_0 = \frac{1}{n}J, E_1, \dots, E_d$, where J is the all–one matrix of order n.

Let $V = \mathbb{C}^X$ denote the vector space over the complex number filed consisting of column vectors whose coordinates are indexed by X with complex entries. We endow V with the Hermitian inner product $\langle\, ,\, \rangle$ defined by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = {}^t\boldsymbol{u}\overline{\boldsymbol{v}},$$

where ${}^t\boldsymbol{u}$ denotes the transpose of $\boldsymbol{u}$, and $\overline{\boldsymbol{v}}$ is the complex conjugate of $\boldsymbol{v}$. We use the following abbreviation $\|\boldsymbol{u}\|^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle$.

It is well-known that V has the following orthogonal decomposition

$$V = E_0V \oplus E_1V \oplus \cdots \oplus E_dV,$$

where $E_0V = \langle \mathbf{1} \rangle$, $\mathbf{1}$ is the all–one vector of size n. Let $\hat{x}$ be the element of V with a 1 in the x-coordinate and 0 in all other coordinates. Moreover, if Γ is a bipartite distance regular graph with bipartitions P_1 and P_2, then $E_dV = \langle \widetilde{\mathbf{1}} \rangle$, where $\widetilde{\mathbf{1}} = \sum_{y \in P_1} \hat{y} - \sum_{z \in P_2} \hat{z}$.

For each i, set

$$A_i = \sum_{j=0}^{d} p_i(j) E_j, \qquad E_i = \frac{1}{n} \sum_{j=0}^{d} q_i(j) A_j.$$

Let $m_i = q_i(0)$ and $\theta_i = p_1(i)$. Then $m_i = \operatorname{rank} E_i$, and $\theta_0, \theta_1, \dots, \theta_d$ are mutually distinct eigenvalues of Γ. By [1, p. 63],

$$\frac{p_i(j)}{k_i} = \frac{q_j(i)}{m_j},$$

where $k_i = |\Gamma_i(x)|$. We also write $k_1 = k$. For each eigenvalue θ_j, let

$$\sigma_i = \sigma_i(\theta_j) = \frac{p_i(j)}{k_i} = \frac{q_j(i)}{m_j}. \tag{2}$$

We call the numbers $\sigma_0, \sigma_1, \dots, \sigma_d$ the *cosine sequence* associated with θ_j.

Let θ be an eigenvalue of Γ and $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the corresponding cosine sequence. By [5, Lemma 2.1],

$$c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i,$$

and

$$\sigma_0 = 1, \qquad \sigma_1 = \frac{\theta}{k}, \qquad \sigma_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}, \qquad \sigma_3 = \frac{\theta^3 - (a_1 + a_2)\theta^2 + (a_1a_2 - k - c_2b_1)\theta + a_2k}{kb_1b_2}.$$

Substituting $\sigma_1, \sigma_2, \sigma_3$ in (2) using the above equalities, we get

$$q_j(1) = \frac{m_j\theta}{k}, \tag{3}$$

$$q_j(2) = \frac{m_j(\theta^2 - a_1\theta - k)}{kb_1}, \tag{4}$$

$$q_j(3) = \frac{m_j(\theta^3 - (a_1 + a_2)\theta^2 + (a_1a_2 - k - c_2b_1)\theta + a_2k)}{kb_1b_2}. \tag{5}$$

For a non-empty set $Y \subseteq X$ and $x \in X$, define $E_0^* = E_0^*(Y)$ to be the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, x)-entry

$$(E_0^*)_{xx} = \begin{cases} 1, & \text{if } x \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

A nonzero vector $\boldsymbol{v} \in E_0^*V$ is said to be *tight* if

$$|\{i | i \in \{0, 1, \ldots, d\}, E_i\boldsymbol{v} = \boldsymbol{0}\}| = w(Y).$$

For $\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}$, let

$$\eta^{(i)}(\boldsymbol{v}) = \frac{{}^t\boldsymbol{v}A_i\overline{\boldsymbol{v}}}{{}^t\boldsymbol{v}\overline{\boldsymbol{v}}}, \qquad \widetilde{A}_i = E_0^*A_iE_0^*.$$

Pick $\boldsymbol{v} \in E_0^*V$. Then

$${}^t\boldsymbol{v}A_i\overline{\boldsymbol{v}} = {}^t\boldsymbol{v}E_0^*A_iE_0^*\overline{\boldsymbol{v}} = {}^t\boldsymbol{v}\widetilde{A}_i\overline{\boldsymbol{v}}.$$

Moreover, $\eta^{(i)}(\boldsymbol{v}) = 0$ for $i > w(Y)$. Furthermore, if $\boldsymbol{v}$ is an eigenvector of $\widetilde{A}_i$, then $\eta^{(i)}(\boldsymbol{v})$ is the eigenvalue of $\widetilde{A}_i$ associated with the eigenvector $\boldsymbol{v}$.

3. Proof of main results

We begin with a useful lemma.

Lemma 3.1. *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with a strongly closed regular subgraph Y of diameter 3, and let $\boldsymbol{v} \in E_0^*V$ be a nonzero vector such that $E_0\boldsymbol{v} = E_d\boldsymbol{v} = \boldsymbol{0}$.*

(i) *For $i \in \{0, 1, \ldots, d\}$,*

$$\frac{\|E_i\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = -\frac{m_i}{kb_1b_2n}(\theta_i^2 - kb_1 - k)(\eta^{(1)}(\boldsymbol{v})\theta_i + b_2) \geq 0,$$

where $k = \theta_0 > \theta_1 > \cdots > \theta_d = -k$ are all distinct eigenvalues of Γ.

(ii) *The following inequalities hold:*

$$-\frac{b_2}{\theta_1} \leq -\sqrt{c_3 - c_2} \leq \eta^{(1)}(\boldsymbol{v}) \leq \sqrt{c_3 - c_2} \leq \frac{b_2}{\theta_1}.$$

(iii) *$\boldsymbol{v}$ is tight if and only if $\eta^{(1)}(\boldsymbol{v}) = \pm\sqrt{c_3 - c_2} = \pm\frac{b_2}{\theta_1}$.*

Proof. (i) By [4, Lemma 8.2] and (3)–(5), we have

$$\begin{aligned}\frac{\|E_i\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} &= \frac{1}{n}\sum_{j=0}^{w(Y)} \eta^{(j)}(\boldsymbol{v})q_i(j) \\ &= \frac{m_i}{n}\left(1 + \eta^{(1)}(\boldsymbol{v})\frac{\theta_i}{k} + \eta^{(2)}(\boldsymbol{v})\frac{\theta_i^2 - k}{kb_1} - (1 + \eta^{(1)}(\boldsymbol{v}) + \eta^{(2)}(\boldsymbol{v}))\frac{\theta_i^3 - (k + c_2b_1)\theta_i}{kb_1b_2}\right).\end{aligned}$$

Observe $\eta^{(2)}(\boldsymbol{v}) = -1$ by $E_d\boldsymbol{v} = \mathbf{0}$, we find

$$\frac{\|E_i\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = -\frac{m_i}{kb_1b_2n}(\theta_i^2 - kb_1 - k)(\eta^{(1)}(\boldsymbol{v})\theta_i + b_2).$$

(ii) For a vector $\boldsymbol{u} \in V$, let $\boldsymbol{u}_Y$ be a vector with xth component

$$(\boldsymbol{u}_Y)_x = \begin{cases} \boldsymbol{u}_x, & \text{if } x \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

Using $E_0\boldsymbol{v} = E_d\boldsymbol{v} = \mathbf{0}$, we see $\langle \boldsymbol{v}, \mathbf{1}\rangle = \langle \boldsymbol{v}, \widetilde{\mathbf{1}}\rangle = 0$. By $\boldsymbol{v} \in E_0^*V$, $\langle \boldsymbol{v}, \mathbf{1}_Y\rangle = \langle \boldsymbol{v}, \widetilde{\mathbf{1}}_Y\rangle = 0$. Suppose η is the second largest eigenvalue of $\widetilde{A}$. Then

$$E_0^*V = \langle \mathbf{1}_Y\rangle \oplus \langle \widetilde{\mathbf{1}}_Y\rangle \oplus W_\eta \oplus W_{-\eta}, \tag{6}$$

where $W_{\pm\eta}$ is the eigenspace of $\widetilde{A}$ with corresponding eigenvalue $\pm\eta$. Then there exist $\boldsymbol{v}_1, \boldsymbol{v}_2 \in E_0^*V$ such that

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2,$$

where $\widetilde{A}\boldsymbol{v}_1 = \eta\boldsymbol{v}_1$, $\widetilde{A}\boldsymbol{v}_2 = -\eta\boldsymbol{v}_2$. Referring to [2, p. 432], $\eta = \sqrt{c_3 - c_2}$.

Note that

$${}^t\boldsymbol{v}A\bar{\boldsymbol{v}} = {}^t\boldsymbol{v}\widetilde{A}\bar{\boldsymbol{v}} = \eta\|\boldsymbol{v}_1\|^2 - \eta\|\boldsymbol{v}_2\|^2,$$

and

$$\eta\|\boldsymbol{v}\|^2 \geq \eta\|\boldsymbol{v}_1\|^2 - \eta\|\boldsymbol{v}_2\|^2 \geq -\eta\|\boldsymbol{v}\|^2$$

reduce to

$$\eta \geq \eta^{(1)}(\boldsymbol{v}) \geq -\eta.$$

By (i), we obtain $(\theta_i^2 - kb_1 - k)(\eta^{(1)}(\boldsymbol{v})\theta_i + b_2) \leq 0$. If $i = 0$ or d, $\theta_i^2 - kb_1 - k = 0$. Then $\theta_i^2 < k^2$ and $\theta_i^2 - kb_1 - k < 0$ for $1 \leq i \leq d-1$. It follows that

$$\eta^{(1)}(\boldsymbol{v})\theta_i + b_2 \geq 0, \quad i = 1, \ldots, d-1. \tag{7}$$

If $\boldsymbol{v} = \boldsymbol{v}_1$ or $\boldsymbol{v}_2$, then $\eta^{(1)}(\boldsymbol{v}) = \pm\eta$, and

$$-\frac{b_2}{\theta_1} \leq \pm\eta \leq \frac{b_2}{\theta_1}.$$

(iii) Using $E_0\boldsymbol{v} = E_d\boldsymbol{v} = \mathbf{0}$, the vector $\boldsymbol{v}$ is tight if and only if there exists a unique $i \in \{1, \ldots, d-1\}$ such that $E_i\boldsymbol{v} = \mathbf{0}$. By (i) and $\theta_i^2 - kb_1 - k < 0$,

$$\eta^{(1)}(\boldsymbol{v})\theta_i + b_2 = 0. \tag{8}$$

Referring to (7), the equality (8) holds if and only if $i = d-1$ or $i = 1$, i.e.,

$$\eta^{(1)}(\boldsymbol{v}) = \pm\eta = \pm\frac{b_2}{\theta_1}. \quad \square$$

Next we start our proof of the main results.

Proof of Theorem 1.1. Referring to (6), we may pick linearly independent eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{|Y|-2}$ of $\widetilde{A}$ with corresponding eigenvalues $\eta, -\eta$. Moreover $\mathbf{1}_Y, \widetilde{\mathbf{1}}_Y$ are eigenvectors of $\widetilde{A}$ with corresponding eigenvalues $c_3, -c_3$, so $\eta^{(1)}(\mathbf{1}_Y) = c_3$ and $\eta^{(1)}(\widetilde{\mathbf{1}}_Y) = -c_3$. Since Y is bipartite distance-regular, we have

$$\mathrm{tr}(\widetilde{A}^2) = c_3|Y|, \quad |Y| = \frac{2(c_3^2 - c_3 + c_2)}{c_2},$$

which imply

$$\sum_{i=1}^{|Y|-2}(\eta^{(1)}(\boldsymbol{u}_i))^2 = c_3|Y| - 2c_3^2 = \frac{2c_3(c_3-1)(c_3-c_2)}{c_2}. \tag{9}$$

Referring to Lemma 3.1(ii),

$$\sum_{j=1}^{|Y|-2}\left(\eta^{(1)}(\boldsymbol{u}_j) - \frac{b_2}{\theta_1}\right)\left(\eta^{(1)}(\boldsymbol{u}_j) + \frac{b_2}{\theta_1}\right) \leq 0. \tag{10}$$

Simplifying (10) using (9), (1) holds. $\square$

Finally we shall characterize the distance-regular graphs with a completely regular strongly closed subgraph $H(3, 2)$.

Let $v_0(t), v_1(t), \ldots, v_d(t), v_{d+1}(t)$ denote the polynomials in $\mathbb{R}[t]$ satisfying $v_0(t) = 1$, and for each i,

$$tv_i(t) = b_{i-1}v_{i-1} + a_iv_i(t) + c_{i+1}v_{i+1}(t), \tag{11}$$

where $b_{-1} = 0$, $c_{d+1} = 1$ and $v_{-1}(t) = 0$. Define

$$\rho_{\mathbf{1}_Y}(t) = \frac{1}{n}\sum_{i=0}^{d}\eta^{(i)}(\mathbf{1}_Y)\frac{v_i(t)}{k_i} \in \mathbb{R}[t]. \tag{12}$$

Proof of Proposition 1.2. By (11) we have

$$v_1(t) = t, \qquad v_2(t) = \frac{1}{c_2}(t^2 - k), \qquad v_3(t) = \frac{1}{c_2c_3}(t^3 - (k + b_1c_2)t).$$

Substituting $v_1(t), v_2(t), v_3(t)$ in (12) using the above equalities, and then simplifying by $k_{i-1}b_{i-1} = k_ic_i$, we get

$$\begin{aligned}
\rho_{\mathbf{1}_Y}(t) &= \frac{1}{n}\sum_{i=0}^{3}\eta^{(i)}(\mathbf{1}_Y)\frac{v_i(t)}{k_i} \\
&= \frac{1}{n}\left(1 + \eta^{(1)}(\mathbf{1}_Y)\frac{t}{k_1} + \eta^{(2)}(\mathbf{1}_Y)\frac{t^2 - k}{k_2c_2} + \eta^{(3)}(\mathbf{1}_Y)\frac{t^3 - (k + b_1c_2)t}{k_3c_2c_3}\right) \\
&= \frac{1}{kb_1b_2n}(\eta^{(3)}(\mathbf{1}_Y)t^3 + b_2\eta^{(2)}(\mathbf{1}_Y)t^2 + (b_1b_2\eta^{(1)}(\mathbf{1}_Y) - (k + b_1c_2)\eta^{(3)}(\mathbf{1}_Y))t + kb_1b_2 - kb_2\eta^{(2)}(\mathbf{1}_Y)).
\end{aligned}$$

Observe

$$\eta^{(i)}(\mathbf{1}_Y) = \frac{{}^t\mathbf{1}_YA_i\overline{\mathbf{1}}_Y}{\|\mathbf{1}_Y\|^2} = \frac{1}{|Y|}|\{(y_1, y_2) \in Y \times Y | \partial(y_1, y_2) = i\}|,$$

so $\eta^{(1)}(\mathbf{1}_Y) = \eta^{(2)}(\mathbf{1}_Y) = 3$, $\eta^{(3)}(\mathbf{1}_Y) = 1$ since Y is 3-cube. Moreover the fact that Y is strongly closed implies $c_2 = 2$, $c_3 = 3$, $b_2 = k - 2$. Hence,

$$\begin{aligned}
\rho_{\mathbf{1}_Y}(t) &= \frac{1}{kb_1b_2n}(t^3 + 3(k-2)t^2 + (3k^2 - 12k + 8)t + k(k-2)(k-4)) \\
&= \frac{1}{kb_1b_2n}(t + k)(t + k - 2)(t + k - 4).
\end{aligned} \tag{13}$$

By [4, Lemma 8.2] and (13),

$$\frac{\|E_i\mathbf{1}_Y\|^2}{\|\mathbf{1}_Y\|^2} = \rho_{\mathbf{1}_Y}(\theta_i)m_i = \frac{m_i}{kb_1b_2n}(\theta_i + k)(\theta_i + k - 2)(\theta_i + k - 4).$$

According to [3, Corollary 5.3], Y is completely regular with covering radius $d - 3$ if and only if $\mathbf{1}_Y$ is tight, so $-k$, $2 - k$ and $4 - k$ are the eigenvalues of Γ. In particular, Γ is a bipartite graph, and $k - 2 = b_1 - 1$ is an eigenvalue of Γ. By the inequality in Theorem 1.1, $\theta_1 \leq k - 2$, so $\theta_1 = k - 2$. By [2, Theorem 4.4.11] Γ is isomorphic to $H(d, 2)$. □

Remark. $H(d, 2)$, Doubled Grassmann graph ($q = 2$ or 3) with diameter 5 and doubled Odd graph on $2m + 1$ points with $m \geq 2$ satisfy the equality in Theorem 1.1.

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